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DYNAMICS OF LAMINAR FREE-CONVECTIVE FLOW OF A NEWTONIAN FLUID BETWEEN VERTICAL PLANE ISOTHERMAL WALLS

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The analytical solution of the unsteady problem on the laminar natural convection of an incompressible viscous fluid in an infinite vertical plane channel whose walls are held at constant and identical temperatures has been obtained.

Keywords: *free convection, plane channel, hydrothermal structure, heat transfer.*

Introduction. Despite the clear physical picture of why free-convective flow is initiated and develops [1], it is difficult, as previously, to identify the basic hydrodynamic and heat-exchange characteristics in connection with problems of integration of fundamental Navier–Stokes equations in the Oberbeck–Boussinesq approximation [2–4]. This is responsible precisely for the sustained interest in synthesizing new solutions [5] based on correct simplifying assumptions, including interest in analyzing the classical problem on free-convective flow in a vertical plane channel under different thermal conditions [6–8]. However, even in this case one cannot obviate the procedure of numerical integration, which substantially diminishes the theoretical and practical significance of the obtained results.

Formulation of the Problem. We consider the problem of initiation, development, and cessation of free-convective flow of an incompressible viscous fluid in a vertical plane channel of unbounded height with the same change in the temperature of lateral walls to a certain constant value that is different from the initial one.

The Oberbeck–Boussinesq equations in dimensionless component form for the plane Cartesian coordinate system $x0z$ that describe free-convective motion of an incompressible viscous medium with an initial temperature t_0 in a rectangular region of half-width h_1 and height h_2 (Fig. 1) and whose lateral walls are held at a temperature t_w are as follows:

$$\frac{\partial V_X}{\partial \theta} + \frac{2}{1+\xi} V_X \frac{\partial V_X}{\partial X} + \frac{2\xi}{1+\xi} V_Z \frac{\partial V_X}{\partial Z} = -\frac{2}{1+\xi}\frac{\partial P}{\partial X} + \frac{4}{(1+\xi)^2}\left(\frac{\partial^2 V_X}{\partial X^2} + \xi^2 \frac{\partial^2 V_X}{\partial Z^2}\right), \tag{1}$$

$$\frac{\partial V_Z}{\partial \theta} + \frac{2}{1+\xi} V_X \frac{\partial V_Z}{\partial X} + \frac{2\xi}{1+\xi} V_Z \frac{\partial V_Z}{\partial Z}$$

$$= -\frac{2\xi}{1+\xi}\frac{\partial P}{\partial Z} + \frac{4}{(1+\xi)^2}\left(\frac{\partial^2 V_Z}{\partial X^2} + \xi^2 \frac{\partial^2 V_Z}{\partial Z^2}\right) + \frac{8}{(1+\xi)^3}\,\mathrm{Gr}\,(T - T^*)\,, \tag{2}$$

$$\frac{\partial T}{\partial \theta} + \frac{2}{1+\xi} V_X \frac{\partial T}{\partial X} + \frac{2\xi}{1+\xi} V_Z \frac{\partial T}{\partial Z} = \frac{1}{\mathrm{Pr}}\frac{4}{(1+\xi)^2}\left(\frac{\partial^2 T}{\partial X^2} + \xi^2 \frac{\partial^2 T}{\partial Z^2}\right), \tag{3}$$

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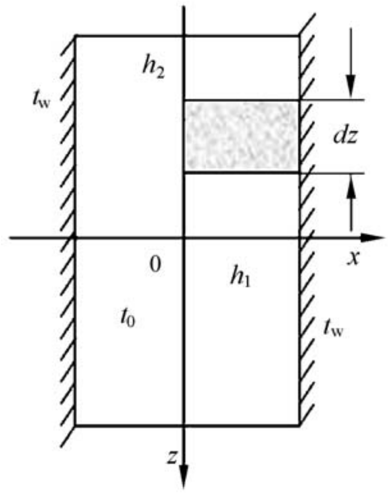

Fig. 1. Computational scheme.

$$\frac{\partial V_X}{\partial X} + \xi \frac{\partial V_Z}{\partial Z} = 0 . \tag{4}$$

When $\xi \to 0$ system (1)–(4) is transformed to the form

$$\frac{\partial V_X}{\partial \theta} = -2 \frac{\partial P}{\partial X} , \tag{5}$$

$$\frac{\partial V_Z}{\partial \theta} + 2V_X \frac{\partial V_Z}{\partial X} = 4 \frac{\partial^2 V_Z}{\partial X^2} + 8\mathrm{Gr}\,(T - T^*) , \tag{6}$$

$$\frac{\partial T}{\partial \theta} + 2V_X \frac{\partial T}{\partial X} = \frac{4}{\mathrm{Pr}} \frac{\partial^2 T}{\partial X^2} . \tag{7}$$

Differentiating (5) with respect to X with account for (4), we obtain

$$\frac{\partial}{\partial X}\left(\frac{\partial V_X}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(\frac{\partial V_X}{\partial X}\right) = -2 \frac{\partial^2 P}{\partial X^2} \equiv 0 , \tag{8}$$

whence $P(X, Z, \theta) = f_1(Z, \theta)X + f_2(Z, \theta)$, where $f_1(Z, \theta)$ and $f_2(Z, \theta)$ are certain functions. However the axial symmetry of the problem yields $f_1(Z, \theta) = 0$; therefore, we have $\partial P/\partial X = 0$, i.e., Y_X = const, and the condition of "adhesion" of the fluid on the wall gives the zero value of this constant ($V_X \equiv 0$); as a result, Eqs. (5)–(7) are transformed to the system

$$\frac{\partial V_Z}{\partial \theta} = 4 \frac{\partial^2 V_Z}{\partial X^2} + 8\mathrm{Gr}\,(T - T^*) , \tag{9}$$

$$\frac{\partial T}{\partial \theta} = \frac{4}{\mathrm{Pr}} \frac{\partial^2 T}{\partial X^2} \tag{10}$$

with obvious boundary conditions

$$V_Z(X, 0) = V_Z(1, \theta) = \partial V_Z(0, \theta)/\partial X = 0\,; \tag{11}$$

$$T(X, 0) = 1\,, \quad T(1, \theta) = \partial T(0, \theta)/\partial X = 0\,. \tag{12}$$

Solution. We apply the integral Laplace transformation to (9)–(12) with respect to the variable θ:

$$\frac{d^2 V_{\mathrm{L}}}{dX^2} - \frac{s}{4} V_{\mathrm{L}} = -2\mathrm{Gr}\left(T_{\mathrm{L}} - \frac{1}{s} T^*\right), \tag{13}$$

$$\frac{d^2 T_{\mathrm{L}}}{dX^2} - \frac{\mathrm{Pr}}{4} s T_{\mathrm{L}} = -\frac{\mathrm{Pr}}{4}\,, \tag{14}$$

$$V_{\mathrm{L}}(1, s) = dV_{\mathrm{L}}(0, s)/dX = 0\,, \tag{15}$$

$$T_{\mathrm{L}}(1, s) = dT_{\mathrm{L}}(0, s)/dX = 0\,. \tag{16}$$

Substitution of the solution of Eq. (14) with boundary conditions (16)

$$T_{\mathrm{L}}(X, s) = \frac{1}{s}\left[1 - \frac{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\, s}\, X\right)}{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\, s}\right)}\right] \tag{17}$$

leads to the differential equation for V_L

$$\frac{d^2 V_{\mathrm{L}}}{dX^2} - \frac{s}{4} V_{\mathrm{L}} = -\frac{2}{s}\mathrm{Gr}\left[1 - T^* - \frac{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\, s}\, X\right)}{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\, s}\right)}\right] \tag{18}$$

with boundary conditions (15). The general solution of the corresponding homogeneous equation for (18) is

$$V_{\mathrm{L}}(X, s) = C_1 \sinh\left(\frac{1}{2}\sqrt{s}\, X\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{s}\, X\right), \tag{19}$$

in which the integration constants $C_1 = C_1(X, s)$ and $C_2 = C_2(X, s)$ have been determined by the method of variation of arbitrary constants from the algebraic system

$$C_1' \sinh\left(\frac{1}{2}\sqrt{s}\, X\right) + C_2' \cosh\left(\frac{1}{2}\sqrt{s}\, X\right) = 0\,,$$

$$C_1' \cosh\left(\frac{1}{2}\sqrt{s}\, X\right) + C_2' \sinh\left(\frac{1}{2}\sqrt{s}\, X\right) = -\frac{4\mathrm{Gr}}{s\sqrt{s}}\left[1 - T^* - \frac{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\, s}\, X\right)}{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\, s}\right)}\right],$$

where $C'_{1,2} = dC_{1,2}(X, s)/dX$, in the form

$$C_1 = -\frac{4\mathrm{Gr}}{s^2}\left\langle -\frac{1}{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\,s}\right)}\left\{\frac{\sinh\left[\frac{1}{2}\sqrt{s}\,(1+\sqrt{\mathrm{Pr}})\,X\right]}{1+\sqrt{\mathrm{Pr}}}\right.\right.$$

$$\left.\left.+\frac{\sinh\left[\frac{1}{2}\sqrt{s}\,(1-\sqrt{\mathrm{Pr}})\,X\right]}{1-\sqrt{\mathrm{Pr}}}\right\} + 2\,(1-T^*)\sinh\left(\frac{1}{2}\sqrt{s}\,X\right)\right\rangle + \tilde{C}_1\,, \tag{20}$$

$$C_2 = \frac{4\mathrm{Gr}}{s^2}\left\langle -\frac{1}{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\,s}\right)}\left\{\frac{\cosh\left[\frac{1}{2}\sqrt{s}\,(1+\sqrt{\mathrm{Pr}})\,X\right]}{1+\sqrt{\mathrm{Pr}}}\right.\right.$$

$$\left.\left.+\frac{\cosh\left[\frac{1}{2}\sqrt{s}\,(1-\sqrt{\mathrm{Pr}})\,X\right]}{1-\sqrt{\mathrm{Pr}}}\right\} + 2\,(1-T^*)\cosh\left(\frac{1}{2}\sqrt{s}\,X\right)\right\rangle + \tilde{C}_2\,, \tag{21}$$

the integration constants $\tilde{C}_1$ and $\tilde{C}_2$ have been found from (15):

$$\tilde{C}_1 = 0\,,\quad \tilde{C}_2 = -\frac{8\mathrm{Gr}}{s^2\cosh\left(\frac{1}{2}\sqrt{s}\right)}\left(1-T^*-\frac{1}{1-\mathrm{Pr}}\right). \tag{22}$$

Expressions (19)–(22) yield

$$V_{\mathrm{L}}(X, s) = -\frac{8\mathrm{Gr}}{s^2}\left[\frac{1}{1-\mathrm{Pr}}\frac{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\,s}\,X\right)}{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\,s}\right)} - 1 + T^* + \left(1-T^*-\frac{1}{1-\mathrm{Pr}}\right)\frac{\cosh\left(\frac{1}{2}\sqrt{s}\,X\right)}{\cosh\left(\frac{1}{2}\sqrt{s}\right)}\right]. \tag{23}$$

We consider first the steady-state component of the inverse transform of the velocity (23), bearing in mind that $s = 0$ (root of multiplicity two). Decomposing (23) into terms and using the Vashchenko–Zakharenko second theorem to identity the inverse transform from the transform representable as the ratio of infinite polynomials in s (the degree of polynomial of the denominator is larger than that of the numerator), we obtain

$$L^{-1}\left[\frac{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\,s}\,X\right)}{s^2\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\,s}\right)}\right] = \theta + \frac{1}{8}\,(X^2-1)\,, \tag{24}$$

$$L^{-1}\left[\frac{\cosh\left(\frac{1}{2}\sqrt{s}\,X\right)}{s^2\cosh\left(\frac{1}{2}\sqrt{s}\right)}\right]=\theta+\frac{1}{8}\,(X^2-1)\,. \tag{25}$$

On the basis of (24) and (25) and in view of $L^{-1}[s^{-2}] = 0$, expression (23) yields

$$L^{-1}\,[V_{\rm L}\,(X, s)] = \lim_{\theta\to\infty} V_Z\,(X, \theta) = -\,8\mathrm{Gr}\,\lim_{\theta\to\infty}\left\{\frac{1}{1-\mathrm{Pr}}\left[\theta+\frac{\mathrm{Pr}}{8}\,(X^2-1)\right]-(1-T^*)\,\theta\right.$$

$$\left.+\left(1-T^*-\frac{1}{1-\mathrm{Pr}}\right)\left[\theta+\frac{1}{8}\,(X^2-1)\right]\right\}=-\,\mathrm{Gr}\,T^*\,(1-X^2)\,.$$

To the physical meaning of the problem there corresponds the condition $V_Z(X, \infty) = 0$, which implies the cessation of flow because of the thermal equilibrium established in the channel, whence $T^* = 0$.

The unsteady components of the inverse transform of velocity are determined by the expressions

$$L^{-1}\left[\frac{\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\,s}\,X\right)}{s^2\cosh\left(\frac{1}{2}\sqrt{\mathrm{Pr}\,s}\right)}\right]=\frac{4}{\pi^3}\,\mathrm{Pr}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\left[\frac{\pi}{2}\,(2n+1)\,X\right]\exp\left[-\frac{\pi^2}{\mathrm{Pr}}\,(2n+1)^2\,\theta\right],$$

$$L^{-1}\left[\frac{\cosh\left(\frac{1}{2}\sqrt{s}\,X\right)}{s^2\cosh\left(\frac{1}{2}\sqrt{s}\right)}\right]=\frac{4}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\left[\frac{\pi}{2}\,(2n+1)\,X\right]\exp\left[-\pi^2\,(2n+1)^2\,\theta\right],$$

whence the transform of velocity is

$$V_Z\,(X, \theta) = V\,(X, \theta) = -\,\frac{32}{\pi^3}\,\mathrm{Gr}\,\frac{\mathrm{Pr}}{1-\mathrm{Pr}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\left[\frac{\pi}{2}\,(2n+1)\,X\right]$$

$$\times\left\{\exp\left[-\frac{\pi^2}{\mathrm{Pr}}\,(2n+1)^2\,\theta\right]-\exp\left[-\pi^2\,(2n+1)^2\,\theta\right]\right\}. \tag{26}$$

When $\mathrm{Pr} \to 1$ there appears an indeterminate form of the $\frac{0}{0}$ type, expanding which according to the l'Hôpital rule, i.e., to

$$(1-\mathrm{Pr})'_{\mathrm{Pr}} = -\,1\,,$$

$$\left\{\exp\left[-\frac{\pi^2}{\mathrm{Pr}}\,(2n+1)^2\,\theta\right]\right\}'_{\mathrm{Pr}}=\exp\left[-\frac{\pi^2}{\mathrm{Pr}}\,(2n+1)^2\,\theta\right]\times\left[-\frac{\pi^2}{\mathrm{Pr}^2}\,(2n+1)^2\,\theta\right],$$

we obtain

$$V(X, \theta) = \frac{32}{\pi} \mathrm{Gr}\, \theta \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos\left[\frac{\pi}{2} (2n+1) X\right] \exp\left[-\pi^2 (2n+1)^2 \theta\right]. \tag{27}$$

The inverse transform of (17) gives the expression for the temperature field

$$T(X, \theta) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos\left[\frac{\pi}{2} (2n+1) X\right] \exp\left[-\frac{\pi^2}{\mathrm{Pr}} (2n+1)^2 \theta\right]. \tag{28}$$

The mean-integral characteristics of the found hydrothermal fields are

$$\overline{V}(\theta) = -\frac{64}{\pi^4} \mathrm{Gr} \frac{\mathrm{Pr}}{1-\mathrm{Pr}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \left\{ \exp\left[-\frac{\pi^2}{\mathrm{Pr}} (2n+1)^2 \theta\right] - \exp\left[-\pi^2 (2n+1)^2 \theta\right] \right\},$$

$$\overline{T}(\theta) = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \exp\left[\frac{-\pi^2}{\mathrm{Pr}} (2n+1)^2 \theta\right], \tag{29}$$

when Pr = 1.

$$\overline{V}(\theta) = \frac{64}{\pi^2} \mathrm{Gr}\, \theta \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \exp\left[-\pi^2 (2n+1)^2 \theta\right], \tag{30}$$

$$\overline{T}(\theta) = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \exp\left[-\pi^2 (2n+1)^2 \theta\right]. \tag{31}$$

Analysis. Computational experiment shows that the medium's velocity in the channel is in proportion to the value of the Grashof number. For fluids with the same viscosity, free-convective flow develops more intensely if the thermal diffusivity of the medium is larger (Fig. 2), whereas for fluids with the same thermal diffusivity, the dependence on kinematic viscosity is inverse. When $\mathrm{Pr} > 1$ the flow velocity is attenuated faster than thermal equilibrium is established; when $\mathrm{Pr} < 1$ the opposite situation is observed (Fig. 3). We easily note than in the case where Pr = 1 the hydrodynamic and thermal fields are attenuated simultaneously.

The relative height $H = z/(2h_1)$ to which the fluid ascends (descends) since the change in the channel-wall temperature is found by solution of the Cauchy problem

$$\frac{dH}{d\theta} = \overline{V}(\theta), \quad H(0) = 0$$

in the form

$$H = \frac{1}{15} \mathrm{Ra},$$

where Ra = GrPr is the Rayleigh number.

Determination of the heat-transfer coefficient α in terms of the temperature gradient on the channel wall does not correspond to the physical meaning in this problem, since the transfer of heat along the coordinate x in the fluid is only by heat conduction. In this connection, from consideration of the thermal balance for a volume element (Fig. 1) under the assumption that the entire specific quantity of heat per unit length, which enters through the wall

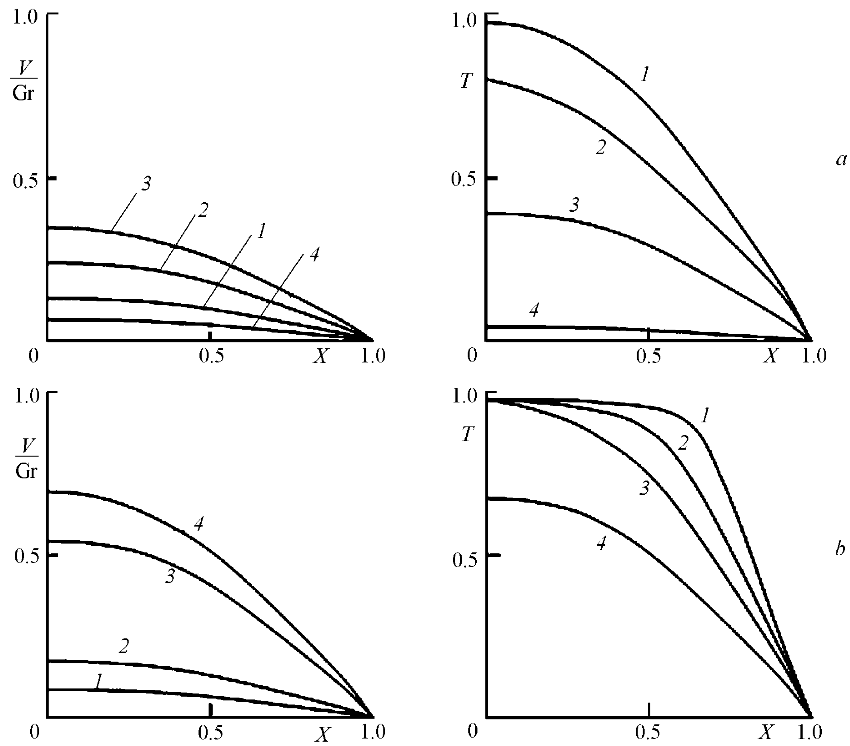

Fig. 2. Structure of hydrothermal fields at Pr = 0.7 (a) and Pr = 7 (b) for different θ: 1) 0.15, 2) 0.03, 3) 0.085, and 4) 0.35.

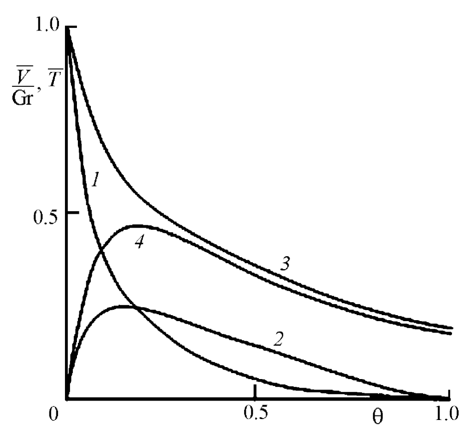

Fig. 3. Average temperature and relative velocity at Pr = 0.7: 1) $\overline{T}$ and 2) $\overline{V}$/Gr; at Pr = 7: 3) $\overline{T}$ and 4) $\overline{V}$/Gr.

$$dQ = -\alpha\,(\tilde{t} - t_{\mathrm{w}})\,dz d\tau\,,$$

goes to heat (cool) it

$$dQ = \rho c_p d\tilde{t} dz h_1\,,$$

we obtain the equation

$$\frac{d\tilde{T}}{d\theta} = -4\,\frac{\mathrm{Nu}}{\mathrm{Pr}}\,\tilde{T}$$

with an obvious initial condition

$$\tilde{T}(0) = 1 ,$$

whose solution is

$$\tilde{T}(\theta) = \exp\left(-4\,\frac{\mathrm{Nu}}{\mathrm{Pr}}\,\theta\right). \tag{32}$$

The equality of the integrals of (31) and (32) between the limits 0–∞ yields that Nu = 3.

Conclusions. The obtained results demonstrate the adequacy of the proposed approach and can be used for evaluating the influence of thermal convection on the operation of various energy-stressed technical devices.

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NOTATION

c_p, specific heat of the fluid at constant pressure, J/(kg·K); g, free-fall acceleration, m^2/sec; $\mathrm{Gr} = gh_1^3\beta\Delta t/\nu^2$, Grashof number; L^{-1}, operator of the inverse one-sided Laplace transformation; $l_0 = 2h_1/(1+\xi)$, characteristic dimension, m; $\mathrm{Nu} = \alpha h_1/\lambda$, Nusselt number; $P = p'/\tilde{p}$, $\tilde{p} = \rho(\nu/l_0)^2$ and $p' = p - p_0$, deviation of pressure from the hydrostatic pressure $p_0 = \rho_0 gz +$ const, Pa; Pr, Prandtl number; s, Laplace transform of θ; $T = (t - t_w)/\Delta t$, $T^* = (t^* - t_w)/\Delta t$, $\tilde{T} = (\tilde{t} - t_w)/\Delta t$, $\Delta t = t_0 - t_w$; t, t^*, and $\tilde{t}$, running, characteristic, and mass-mean temperatures, ^{o}C; $V_X = v_x/\tilde{v}$, $V_Z = v_z/\tilde{v}$; $\tilde{v} = \nu/l_0$, characteristic velocity, m/sec; v_x and v_z, horizontal and vertical velocity components, m/sec; $X = x/h_1$ and $Z = z/h_1$; β, coefficient of volumetric expansion of the fluid, K^{-1}; λ, thermal conductivity of the fluid, W/(m·K); ν, kinematic viscosity of the fluid, m^2/sec; $\theta = \nu\tau l_0^2$; ρ, density of the fluid, kg/m^3; $\rho_0 \approx \rho$, ρ_0, density of the fluid at t_0, kg/m^3; τ, time, sec; $\xi = h_1/h_2$.

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